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COMMENT

On the solution for a vacuum Bianchi type-III model with a cosmological constant

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Abstract. We should like to point out that the solution given by Moussiaux *et al* and MacCallum *et al* has been (in principle) independently obtained by Stewart and Ellis. In addition we present new exact Bianchi–Kantowski–Sachs solutions of type III with dust and a cosmological constant, which are generalisations of the vacuum solutions.

The particular solution for a 'diagonal' vacuum Bianchi type-III model with a non-vanishing cosmological constant Λ, given by Moussiaux *et al* (1981), has been shown in a recent comment by MacCallum *et al* (1982) to be contained in a general solution given by Cahen and Defrise (1968), so this solution is not entirely new. However, these authors have overlooked the paper by Stewart and Ellis (1968), who obtained an integral solution for the vacuum Bianchi type-III model with a cosmological constant and a homogeneous electromagnetic field (see equation (4.12) of the quoted paper). As pointed out by Stewart and Ellis the integration can be performed analytically if $c = 0$ (c = constant), or $\Lambda = 0$. In addition the connection with the Reissner–Nordström-type solutions is mentioned.

The field equations of the general theory of relativity for spatially homogeneous, but anisotropic space–times have been investigated by many authors over the last three decades since the basic work of Taub (1951). One would like to conjecture rather that all integrable cases have been found during this period. However, in a systematic investigation of the combined Einstein–Maxwell equations for all spatially homogeneous models we have found that many cases had not been approached until now. The mistake made by many workers in the field is that they start with much too specialised field equations in the hope of finding simple solutions. In this comment we should like to present exact dust solutions with a cosmological constant for the Bianchi type-III models and the related Kantowski–Sachs space–times. It is shown that the solutions given by Stewart and Ellis (1968), Cahen and Defrise (1968) and Moussiaux *et al* (1981) are special cases of these new solutions.

The dust solutions can be easily obtained from

$$2R_1''/R_1-(R_1'/R_1)^2-4q^2\delta+[\varepsilon(\gamma-1)-\Lambda]R_1^2=0 \tag{1a}$$

$$\frac{R_3'}{R_3}\frac{R_1'}{R_1}=\left(\frac{R_1'}{R_1}\right)'+\tfrac{1}{2}\varepsilon\gamma R_1^2 \tag{1b}$$

$$R_1'/R_1=R_2'/R_2 \qquad \mathrm{d}t=R_1\mathrm{d}\tau \qquad (\ )'=\mathrm{d}/\mathrm{d}\tau \tag{1c}$$

where $\delta = 1$ corresponds to the Bianchi type-III model and $4q^2\delta = -1$ and $R_1 = R_2$ to the Kantowski–Sachs model (see Lorenz 1982 for details), $p = (\gamma - 1)\varepsilon$ ($\gamma = 1$ for dust) and $\varepsilon = \varepsilon_0^2/(R_1^2 R_3)^\gamma$, $\varepsilon_0^2 = \text{constant}$.

We obtain in the case of $\delta = 1$

(i) $\Lambda > 0$

$$R_1 = R_2 = \pm 2q(3/\Lambda)^{1/2}(\sinh 2q\tau)^{-1} \tag{2a}$$

$$R_3 = (a + b\varepsilon_0^2/8q^2)\coth 2q\tau + (\varepsilon_0^2/8q^2)(2q\tau \coth 2q\tau - 1). \tag{2b}$$

(ii) $\Lambda < 0$

$$R_1 = R_2 = 2q(3/|\Lambda|)^{1/2}(\cosh 2q\tau)^{-1} \tag{2c}$$

$$R_3 = (a + b\varepsilon_0^2/8q^2)\tanh 2q\tau + (\varepsilon_0^2/8q^2)(2q\tau \tanh 2q\tau - 1). \tag{2d}$$

The Kantowski–Sachs solutions are given by

$\Lambda > 0$

$$R_1 = \pm(3/\Lambda)^{1/2}(\sin \tau)^{-1} \tag{3a}$$

$$R_3 = (a + \tfrac{1}{2}b\varepsilon_0^2)\cot \tau + \tfrac{1}{2}\varepsilon_0^2(\tau \cot \tau - 1). \tag{3b}$$

a and b are constants of integration. For $\varepsilon_0^2 = 0$ our solutions (2a)–(2d) reduce to the pure vacuum solutions of the authors quoted above. Finally we should like to point out that exact magnetic dust solutions with (or without) a cosmological constant have recently been found by us (Lorenz 1982). This paper also contains many other new solutions (and known results in a unique parametrisation) for the Bianchi–Kantowski–Sachs models.

Note added in proof. The most general solution of (1) with $(\Lambda, c, \varepsilon_0^2) \neq (0, 0, 0)$, where c is a constant of integration (see Moussiaux *et al* 1981), can be obtained in terms of elliptic functions and divides into two distinct types. We find

(i)

$$R_1 = c\frac{1 - \text{cn}(\alpha\tau, k_1)}{\alpha^2 + e + (\alpha^2 - e)\,\text{cn}(\alpha\tau, k_1)} \tag{4a}$$

$$\begin{aligned} R_3 = {} & \frac{2\alpha^3 c\,\text{sn}(\alpha\tau, k_1)\,\text{dn}(\alpha\tau, k_1)}{(1 - \text{cn}(\alpha\tau, k_1))[\alpha^2 + e + (\alpha^2 - e)\,\text{cn}(\alpha\tau, k_1)]}\left\{a + \frac{\varepsilon_0^2}{8\alpha^7}\left[\frac{2ek(\alpha^2 - e)}{\alpha k_1^2}\tau\right.\right. \\ & - 8e^2\left(E(2\varphi, k_1) + \frac{\text{sn}(2\alpha\tau, k_1)\,\text{dn}(2\alpha\tau, k_1)}{1 - \text{cn}(2\tau, k_1)}\right) + \left(\frac{(\alpha^2 + e)^2}{1 - k_1^2} + \frac{(\alpha^2 - e)^2}{k_1^2}\right)E(\varphi, k_1) \\ & + 4e(\alpha^2 + e)\frac{\text{dn}(\alpha\tau, k_1)}{\text{sn}(\alpha\tau, k_1)} - \left(\frac{(\alpha^2 + e)^2}{1 - k_1^2}k_1^2 - (\alpha^2 - e)^2\right)\frac{\text{sn}(\alpha\tau, k_1)\,\text{cn}(\alpha\tau, k_1)}{\text{dn}(\alpha\tau, k_1)} \\ & \left.\left. - 8e^2 k_1^2\frac{\text{sn}(\alpha\tau, k_1)}{\text{dn}(\alpha\tau, k_1)} + b\right]\right\} \end{aligned} \tag{4b}$$

(ii)

$$R_1 = c\frac{1 - \text{cn}(\beta\tau, k_2)}{e(1 - \text{cn}(\beta\tau, k_2)) + \beta^2(1 + \text{dn}(\beta\tau, k_2))} \tag{4c}$$

$$\begin{aligned} R_3 = {} & \frac{\beta^3\,\text{sn}(bt, k_2)[1 + \text{dn}(\beta\tau, k_2) - k_2^2(1 - \text{cn}(\beta\tau, k_2))]}{(1 - \text{cn}(\beta\tau, k_2))[e(1 - \text{cn}(\beta\tau, k_2)) + \beta^2(1 + \text{dn}(\beta\tau, k_2))]} \\ & \times\left[a + \frac{\varepsilon_0^2}{2\beta^6}\int\left(\frac{e\,\text{sn}^2(\beta\tau/2, k_2) + \beta^2}{\text{sn}(\beta\tau, k_2)\,\text{dn}^2(\beta\tau/2, k_2)}\right)^2 \text{d}\tau\right] \end{aligned} \tag{4d}$$

where

$$\alpha^4 = e(3e - 2k), \qquad 2\beta^2 = k - 3e + (1 - 3e^2 + 2ek)^{1/2} \tag{4e}$$

$$4k_1^2 = 2 + (k - 3e)\alpha^{-2}, \qquad k_2^2 = -1 + (k - 3e)\beta^{-2} \tag{4f}$$

$$k = -4q^2\delta, \qquad e^2(k - e) = c^2\Lambda/3 \tag{4g}$$

and $E(2\varphi, k_1)$, $E(\varphi, k_1)$ are elliptic integrals of the second kind with $\varphi = \mathrm{cn}^{-1}(\alpha\tau, k_1)$.

A detailed derivation and discussion of these new solutions will be given in a forthcoming paper (Lorenz 1982).

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